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Aequationes Mathematicae

On an alternative d'Alembert's equation

Roman Ger

Dedicated to Professor Karol Baron on the occasion of his 70th birthday.

Abstract. Roger Cuculière [*Problem 11998*, The American Mathematical Monthly **124** no. 7 (2017)] has posed the following problem: *Find all continuous functions $f:\mathbb{R}\longrightarrow\mathbb{R}$ that satisfy $f(z)\le 1$ for some nonzero real number z and*

$$f(x)^2+f(y)^2+f(x+y)^2-2f(x)f(y)f(x+y)=1 \tag{C}$$

for all real numbers x and y. We present the general Lebesgue measurable solution of (C) in the class of complex valued functions defined on the real line. Moreover, applying the invariant ideals method, we shall discuss a corresponding alternative d'Alembert equation

$$f(x+y)\ne f(x-y)\Longrightarrow f(x+y)+f(x-y)=2f(x)f(y), \tag{CA}$$

stemming from Eq. (C) in the class of scalar valued functions defined on suitable groups. Equations (CA) seems to be of interest on its own.

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1. Introduction

The following question was asked by a French mathematician Roger Cuculière [*Problem 11998*, The American Mathematical Monthly **124** no. 7 (2017)]:

Find all continuous functions $f:\mathbb{R}\longrightarrow\mathbb{R}$ that satisfy $f(z)\le 1$ for some nonzero real number z and

$$f(x)^2+f(y)^2+f(x+y)^2-2f(x)f(y)f(x+y)=1 \tag{C}$$

for all real numbers x and y.

Let $(G,+)$ be a semigroup (not necessarily commutative). Equation (C) may be rewritten in the form

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$$[f(x+y) - f(x)f(y)]^2 = \left(1 - f(x)^2\right)\left(1 - f(y)^2\right), \tag{1}$$

in the class of all complex valued functions f defined on G. Assuming that $(G,+)$ admits a neutral element 0 on setting $c := f(0)$ and $x = y = 0$ in (1) we infer that

$$2(c-1)^2\left(c + \frac{1}{2}\right) = 2c^3 - 3c^2 + 1 = 0, \quad \text{i.e.} \quad c \in \left\{-\frac{1}{2}, 1\right\}.$$

If we had $c = -\frac{1}{2}$ then putting $y = 0$ in (C) we would get $f(x)^2 \equiv \frac{1}{4}$ which, because of (1), forces the equality

$$[f(x+y) - f(x)f(y)]^2 = \frac{9}{16}, \quad x, y \in G.$$

Hence $S := \{x \in G : f(x) = -\frac{1}{2}\}$ yields a submonoid of G.

All these observations lead to the following two propositions.

Proposition 1. *If $(G,+;0)$ is a monoid and $f : G \longrightarrow \mathbb{C}$ satisfies Eq. (1), then $f(0) = 1$ provided that* card $f(G) \geq 3$. *If $f(0) \neq 1$, then either $f(x) \equiv -\frac{1}{2}$ or there exists a proper submonoid $(S,+;0)$ of $(G,+;0)$ such that*

$$f(x) = \begin{cases} -\frac{1}{2} & \mathit{for} \;\; x \in S, \\ \frac{1}{2} & \mathit{for} \;\; x \in G \backslash S =: S' \end{cases}$$

with $S' + S' \subset S$, $S + S' \subset S'$, and $S' + S \subset S'$; in particular, $(S,+)$ yields a subgroup of index 2 of the group $(G,+)$.

Conversely, a constant function $f = -\frac{1}{2}$ as well as any function f of the type described above yields a solution to Eq. (1).

Example. Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ stand for the set of all nonnegative integers. Then the function $f : \mathbb{N}_0 \longrightarrow \mathbb{R}$ given by the formula

$$f(x) = \begin{cases} -\frac{1}{2} & \mathit{for} \;\; x \in 2\mathbb{N}_0, \\ \frac{1}{2} & \mathit{for} \;\; x \in 2\mathbb{N}_0 + 1 \end{cases}$$

satisfies Eq. (1).

Proposition 2. *If $(G,+;0)$ is a monoid and $f : G \longrightarrow \mathbb{C}$ satisfies Eq. (1) with $f(0) = 1$ and* card $f(G) \leq 2$, *then either $f(x) \equiv 1$ or there exists a proper submonoid $(S,+;0)$ of $(G,+;0)$ such that*

$$f(x) = \begin{cases} 1 & \mathit{for} \;\; x \in S, \\ c_0 & \mathit{for} \;\; x \in G \backslash S =: S' \end{cases}$$

with $c_0 \in \mathbb{C} \backslash \{1\}$ and $S' + S' \subset S$, $S + S' \subset S' \neq \emptyset$, $S' + S \subset S'$, (in particular, $(S,+)$ yields a subgroup of index 2 of the group $(G,+)$),
or

$$f(x) = \begin{cases} 1 & \mathit{for} \;\; x \in S, \\ -\frac{1}{2} & \mathit{for} \;\; x \in S' \end{cases}$$

with $S + S' \subset S' \neq \emptyset$, $S' + S \subset S'$.

Conversely, the constant function $f = 1$ as well as any function f of the type described above yields a solution to Eq. (1).

In what follows we shall assume that $(G, +)$ is an Abelian group and $f : G \longrightarrow \mathbb{C}$ is a solution of Eq. (1) with card $f(G) \geq 3$. Then, by Proposition 1, we have $f(0) = 1$. Moreover, the set

$$Z := \{x \in G : \ f(x) = 1\}$$

forms an additive subgroup of $(G, +)$. In fact, $Z \neq \emptyset$ because $0 \in Z$ and $(Z, +)$ yields a submonoid of $(G, +)$ because of (1). Setting $y = -x$ in (1) we infer that

$$1 + f(x)^2 + f(-x)^2 - 2f(x)f(-x) = 1, \ x \in G,$$

which proves that f is even.

Consequently,

$$(Z, +) \text{ yields a subgroup of } (G, +).$$

Now, replacing y by $-y$ in (1) and subtracting the resulting equation from (1) side by side we arrive at the following alternative d'Alembert functional equation:

$$f(x + y) \neq f(x - y) \Longrightarrow f(x + y) + f(x - y) = 2f(x)f(y) \tag{CA}$$

valid for all $x, y \in G$.

In particular, on setting $y = x$ in (CA) we get

$$f(2x) \in \{1, 2f(x)^2 - 1\} \text{ for all } x \in G. \tag{2}$$

We shall discuss Eq. (CA) separately, in Sect. 4 below.

2. A tool: invariant ideals

In the sequel we are going to consider a restricted version of d'Alembert's functional equation (C), assuming that it is satisfied for *almost all* pairs of elements of a (possibly noncommutative) group $(X, +)$, i.e. for all except those that belong to a "small" set from a proper invariant set ideal in X^2. Recall that a proper nonempty subfamily $\mathcal{J} \subset 2^X$ is called a *proper invariant ideal* (p.i. ideal) in X whenever it is closed under finite set-theoretical unions, hereditary with respect to descending inclusions and such that jointly with any member E of $\mathcal{J}$ it contains the family $\{x - E : x \in X\}$.

We say that a property holds $\mathcal{J}$-almost everywhere ($\mathcal{J}$-a.e.) on X provided that it is satisfied on the whole X except for some set $E \in \mathcal{J}$.

For a subset $N \subset X^2$ we define the vertical section of N through the point $x \in X$ by the following formula

$$N[x] := \{y \in X : \ (x, y) \in N\}.$$

If $\widehat{\mathcal{J}}$ is a p.i. ideal in X^2 and the following Fubini type condition is satisfied:

$$\text{for all } N \in \widehat{\mathcal{J}} \text{ the section } N[x] \text{ falls into } \mathcal{J},\ \mathcal{J}\text{-a.e. on } X,$$

then $\widehat{\mathcal{J}}$ is termed to be *conjugate* with $\mathcal{J}$.

To justify the use of a seemingly too abstract machinery let us keep in mind the undermentioned important examples of p.i. ideals:

- Let $(X, +)$ be a group of infinite order; the family of all finite subsets of X yields a p.i. ideal.
- Let $(X, +)$ be a Baire topological group; the family of all first category subsets of X yields a p.i. ideal.
- Let $(X, +)$ be a metric topological group with invariant metric and infinite diameter; the family of all subsets of X that are metrically bounded yields a p.i. ideal.
- Let $(X, +)$ be an Abelian locally compact topological group with a Haar measure h such that $h(X)$ is infinite; the family of all measurable subsets of X having finite measure and all their subsets yields a p.i. ideal.
- Let $(X, +)$ be a locally compact topological group with a completed Haar measure h; the family of all sets $E \subset X$ such that $h(E) = 0$ yields a p.i. ideal.
- Let $\mathcal{J}$ stand for a p.i. ideal in a group $(X, +)$. Then the families

  $$\Pi(\mathcal{J}) := \{N \subset X^2 : N \subset (X \times E) \cup (E \times X) \text{ for some } E \in \mathcal{J}\}$$

  and

  $$\Omega(\mathcal{J}) := \{N \subset X^2 : N[x] \in \mathcal{J} \text{ for } \mathcal{J}-\text{almost all } x \in X\}$$

  yield p.i. ideals in the product group $(X^2, +)$. Both of them are conjugate with $\mathcal{J}$; moreover, $\Pi(\mathcal{J})$ is the smallest (in the sense of inclusion) and $\Omega(\mathcal{J})$ is the maximal one among all p.i. ideals in $(X^2, +)$ congugate with $\mathcal{J}$.
- Let $(X, +)$ be a group. Given a nonempty collection $\mathcal{R}$ of subsets of X, the family $\mathcal{J}(\mathcal{R})$ of all finite unions of the sets

  $$x + [A \cup (-A)] + y, \quad A \in \mathcal{R},\ x, y \in X,$$

  and all their subsets yields the smallest invariant ideal of subsets of X among those containing the family $\mathcal{R}$. An ideal $\mathcal{J}(\mathcal{R})$ becomes a p.i. ideal provided that no finite union of sets of that form coincides with X. If that is the case, this p.i. ideal is termed to be *generated* by the family $\mathcal{R}$.

3. The main results

The following theorem offers a solution of an abstract version of Cuculière's problem for complex valued functions defined on suitable groups. Jointly with the succeeding corollaries it also gives a solution of the original Cuculière's problem (cf. the Introduction).

Theorem 1. *Let $(G,+)$ be an Abelian uniquely 2-divisible group and let $\mathcal{J}_1, \mathcal{J}_2$ be two conjugate p.i. ideals in G and G^2, respectively, enjoying the following properties:*

- $U \in \mathcal{J}_1$ *implies* $2U, \frac{1}{2}U \in \mathcal{J}_1$,
- $N \in \mathcal{J}_2$ *implies* $\varphi(N) \in \mathcal{J}_2$, *where* $\varphi(x,y) := (x+y, x-y), (x,y) \in G^2$.

Let further $f : G \longrightarrow \mathbb{C}$ be a solution to equation

$$f(x)^2 + f(y)^2 + f(x+y)^2 - 2f(x)f(y)f(x+y) = 1,\ (x,y) \in G^2, \qquad \text{[C]}$$

such that $\operatorname{card} f(G) \geq 3$ *and*

$$\{(x,y) \in G^2 : f(x+y) = f(x-y)\} \in \mathcal{J}_2.$$

Then there exists a complex valued exponential function m on G such that

$$f(x) = \frac{m(x) + m(-x)}{2} \quad \text{for all } x \in G; \tag{3}$$

m is unique except that it can be replaced by $m \circ (-id)$.

Conversely, each function $f : G \longrightarrow \mathbb{R}$ defined by formula (3) yields a solution to Eq. (C).

Proof. (Necessity.) We already know that f yields a solution to (CA). Applying Theorem 1 from Adamaszek's paper [2] we infer that f has to be $\mathcal{J}_1$-almost everywhere equal to a solution of d'Alembert's equation (A), i.e. there exist a member E of the ideal $\mathcal{J}_1$ and a function $g : G \longrightarrow \mathbb{C}$ such that

$$\begin{aligned} &g(x+y) + g(x-y) \\ &\quad = 2g(x)g(y) \quad \text{for all } x,y \in G \qquad \text{and} \qquad f(x) = g(x) \quad \text{for all } x \in G \backslash E. \end{aligned}$$

By a simple change of variables equations (CA) and (A) may equivalently be written in the form

$$f(s) \neq f(t) \Longrightarrow f(s) + f(t) = 2f\left(\frac{s+t}{2}\right) f\left(\frac{s-t}{2}\right),\ s,t \in G, \tag{*}$$

and

$$g(s) + g(t) = 2g\left(\frac{s+t}{2}\right) g\left(\frac{s-t}{2}\right),\ s,t \in G, \tag{**}$$

respectively. □

Fix arbitrarily an $s \in G$. By choosing a t off the set $E_s := E \cup (2E - s) \cup (s - 2E) \in \mathcal{J}_1$ we get

$$f(t) = g(t),\ f\left(\frac{s+t}{2}\right) = g\left(\frac{s+t}{2}\right),\ \text{and } f\left(\frac{s-t}{2}\right) = g\left(\frac{s-t}{2}\right),$$

whence, on account of $(*)$ and $(**)$, we infer that

$$f(s) \neq g(t) \Longrightarrow f(s) + g(t) = 2g\left(\frac{s+t}{2}\right) g\left(\frac{s-t}{2}\right) = g(s) + g(t),\ t \in G \backslash E_s,$$

i.e.

$$f(s) \neq g(t) \Longrightarrow f(s) = g(s),\ t \in G \backslash E_s. \tag{4}$$

Put $c := f(s)$ and $T := \{t \in G : g(t) \neq c\}$. If we had $T \in \mathcal{J}_1$, then for every $x \in G$ and $t \in G$ taken off the set

$$T_x := T \cup (2T - x) \cup (x - 2T) \in \mathcal{J}_1$$

we would have

$$g(t) = g\left(\frac{x+t}{2}\right) = g\left(\frac{x-t}{2}\right) = c.$$

Thus, in view of $(**)$, $g(x) + c = 2c^2$ whence

$$g(x) \equiv 2c^2 - c = \text{const} =: \alpha.$$

This gives $f(x) = \alpha$ for all $x \in G \backslash E$. Fix arbitrarily a $u \in G$ and choose a v off the set $E \cup (2E - u) \cup (u - 2E) \in \mathcal{J}_1$ to obtain

$$v \notin E,\ \ \frac{u+v}{2} \notin E,\ \text{ and }\ \frac{u-v}{2} \notin E.$$

Now, an appeal to $(*)$ yields

$$f(u) \neq \alpha \Longrightarrow f(u) + \alpha = 2\alpha^2,$$

whence

$$f(u) \in \{\alpha, 2\alpha^2 - \alpha\} \quad \text{for every} \quad u \in G.$$

This leads to a contradiction:

$$3 \leq \operatorname{card} f(G) \leq 2.$$

Therefore $T \notin \mathcal{J}_1$, which guarantees the existence of a $t \in T \backslash E_s$. Consequently, relation (4) implies the equality $f(s) = g(s)$ whence $f = g$ because of the unrestricted choice of an element s from G. Now, an appeal to Pl. Kannappan's Theorem 2 from his paper [5], proves that there exists a unique function $m : G \longrightarrow \mathbb{C}$ [except that it can be replaced by $m \circ (-id)$] such that

$$m(x+y) = m(x)m(y) \text{ for all } x, y \in G \quad \text{and} \quad f(x) = g(x)$$
$$= \frac{m(x) + m(-x)}{2} \quad \text{for all } x \in G,$$

as claimed.

This finishes the proof because the detailed verification of the sufficiency reduces to a mechanical calculation.

In what follows the symbol ℓ_k denotes the k-dimensional Lebesgue measure, $k \in \{1, 2\}$, whereas $\mathcal{J}_k$ will stand for the p.i. ideal of all Lebesgue nullsets in

$\mathbb{R}^k$, $k \in \{1, 2\}$. Plainly, the ideals $\mathcal{J}_1$ and $\mathcal{J}_2$ are conjugate and satisfy the conditions occurring in the statement of Theorem 1.

Theorem 2. *A Lebesgue measurable function $f : \mathbb{R} \longrightarrow \mathbb{C}$ yields a solution of the equation*

$$f(x)^2 + f(y)^2 + f(x+y)^2 - 2f(x)f(y)f(x+y) = 1 \tag{C}$$

for all $x, y \in \mathbb{R}$, if and only if either $f(x) = -\frac{1}{2}$, $x \in \mathbb{R}$, or

$$f(x) = \begin{cases} -\frac{1}{2} & for\ \ x \in S, \\ \frac{1}{2} & for\ \ x \in G \setminus S =: S' \end{cases}$$

where $(S, +)$ is a Lebesgue measurable subgroup of the group $(\mathbb{R}, +)$ with $\ell_1(S) = 0$, or

$$f(x) = \cos bx \cosh ax + i \sin bx \sinh ax,\ x \in \mathbb{R}, \tag{5}$$

where $a, b \in \mathbb{R}$ stand for arbitrary constants.

(Note that taking $a = b = 0$ we get another constant solution $f(x) = 1$, $x \in \mathbb{R}$.)

Proof. (Sufficiency.) Obviously, a constant function $f = -\frac{1}{2}$ yields a solution to (C). A straightforward verification shows that the two-valued function f defined in the statement of Theorem 2 satisfies Eq. (C) as well. Recall that so does each function of the form (3) with an exponential mapping m; in our present case, the function f given by formula (5) has the form (3) with $m(x) = e^{(a+bi)x}, x \in \mathbb{R}$.

(Necessity.) Assume that a Lebesgue measurable function $f : \mathbb{R} \longrightarrow \mathbb{C}$ yields a solution of Eq. (C). Let us distinguish two cases:

$$\text{(a)} \quad \text{card} f(\mathbb{R}) \leq 2 \qquad \text{and} \qquad \text{(b)} \quad \text{non(a)}.$$

Assume (a). If $f(0) \neq 1$ then, in view of the fact that the group $(\mathbb{R}, +)$ does not admit subgroups of index 2, Proposition 1 implies that $f(x) \equiv -\frac{1}{2}$. If $f(0) = 1$, then from Proposition 2 we infer that either $f(x) \equiv 1$ or

$$f(x) = \begin{cases} 1 & \text{for } x \in S, \\ -\frac{1}{2} & \text{for } x \in S' \end{cases}$$

where $S := f^{-1}(\{1\})$ is a Lebesgue measurable proper subgroup of the group $(\mathbb{R}, +)$. If the Lebesgue measure of $S = S - S$ were positive, then by the Theorem of Steinhaus (see [6]) S would contain a neighbourhood of zero, forcing the equality $S = \mathbb{R}$ and contradicting the fact that $(S, +)$ is proper.

Assume (b). Then we have $f(0) = 1$, f is even, and Eq. (CA) is satisfied.

Due to the measurability of f the set $N_0 := \{(s, t) \in \mathbb{R}^2 : f(s) = f(t)\}$ is

Lebesgue measurable in $\mathbb{R}^2$. Note that the set $N := \{(x, y) \in \mathbb{R}^2 : f(x+y) = f(x-y)\}$ coincides with the image of N_0 through a diffeomorphism $\Phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by the formula

$$\Phi(s,t) = \left(\frac{s+t}{2}, \frac{s-t}{2}\right), \quad (s,t) \in \mathbb{R}^2 .$$

Therefore N is ℓ_2-measurable as well and $\ell_2(N) = \frac{1}{2}\ell_2(N_0)$. We are going to show that

$$N = \{(x, y) \in \mathbb{R}^2 : f(x+y) = f(x-y)\} \in \mathcal{J}_2 . \tag{6}$$

Indeed, otherwise, we would have $0 < \ell_2(N) = \frac{1}{2}\ell_2(N_0)$ and, by Fubini's Theorem, there exists an $s_0 \in \mathbb{R}$ such that the vertical section $N(s_0) := \{t \in \mathbb{R} : f(t) = f(s_0)\}$ of the set N_0 passing through s_0 is of positive ℓ_1-measure. Denote by α the number $f(s_0)$ and fix arbitrarily x and y from $N(s_0)$. An appeal to Eq. (C) jointly with the evenness of f gives now the equality

$$2\alpha^2 + f(x-y)^2 - 2\alpha^2 f(x-y) = 1 ,$$

which implies that $f(x-y) \in \{1,\, 2\alpha^2-1\}$. In other words, $f(D) \subset \{1,\, 2\alpha^2-1\}$ where $D := N(s_0) - N(s_0)$. Recalling that $N(s_0)$ has positive measure we read from Steinhaus' theorem (see [6]) that $D \supset (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$.

Note that due to the measurability of f the group $Z = \{x \in \mathbb{R} : f(x) = 1\}$ is measurable. It has measure zero; indeed, otherwise, applying the Steinhaus Theorem again, $Z = \mathbb{R}$ contradicting the fact that f admits at least three different values. Consequently, we get

$$f(x) = 2\alpha^2 - 1 \text{ for all } x \in (-\varepsilon, \varepsilon)\setminus Z \quad (\text{i.e. for } \ell_1\text{–almost all } x \in (-\varepsilon, \varepsilon)).$$

With the aid of the duplication formula (2), on taking an

$$x \in \left(-\frac{1}{2}\varepsilon, \frac{1}{2}\varepsilon\right) \setminus \left(Z \cup \frac{1}{2}Z\right) = \left(-\frac{1}{2}\varepsilon, \frac{1}{2}\varepsilon\right) \setminus \frac{1}{2}Z,$$

we obtain the (in)equalities

$$1 \neq f(2x) = 2\alpha^2 - 1 \quad \text{and} \quad 2\alpha^2 - 1 = 2(2\alpha^2-1)^2 - 1,$$

whence $2\alpha^2 - 1 = -\frac{1}{2}$. Thus

$$f(x) = -\frac{1}{2} \ \text{ for all } \ x \in (-\varepsilon, \varepsilon) \setminus Z$$

and, consequently, for every $t \in (-2\varepsilon, 2\varepsilon) \setminus Z$ one has $\frac{1}{2}t \in (-\varepsilon, \varepsilon) \setminus \frac{1}{2}Z \subset (-\varepsilon, \varepsilon) \setminus Z$, which implies that

$$f(t) = 2f\left(\frac{1}{2}t\right)^2 - 1 = -\frac{1}{2} \ \text{ for all } \ t \in (-2\varepsilon, 2\varepsilon) \setminus Z.$$

Hence, by induction, for every positive integer n we obtain

$$f(t) = -\frac{1}{2} \ \text{ for all } \ t \in (-2^n\varepsilon, 2^n\varepsilon) \setminus Z.$$

This says nothing else but the equality

$$f(x) = \begin{cases} -\frac{1}{2} & \text{for} \quad x \in \mathbb{R} \setminus Z \\ 1 & \text{for} \quad x \in Z \ , \end{cases}$$

contradicting the fact that card $f(\mathbb{R}) \geq 3$.

So, under the assumptions we deal with, condition (6) holds true. Thus, Theorem 1 may be applied to get the equality

$$f(x) = \frac{m(x) + m(-x)}{2} \quad \text{for all } x \in \mathbb{R}, \tag{7}$$

where $1 \neq m \neq 0$ is a complex valued exponential function on $\mathbb{R}$. Then m and $m \circ (-id)$ are different exponential functions and an appeal to Theorem 3.18 (f) from H. Stetkær's monograph [7] shows that the Lebesgue measurability of m is inherited from that of f. On account of Theorem 4 in Section 5.1 of the monograph by Aczél and Dhombres [1] gives now a representation of m in the form

$$m(x) = e^{cx}, \ x \in \mathbb{R}, \quad \text{where} \quad c = a + bi \ \text{ is a complex number.}$$

Consequently,

$$f(x) = \frac{e^{ax} + e^{-ax}}{2} \cos bx + i(\sin bx) \frac{e^{ax} - e^{-ax}}{2} \quad \text{for all } x \in \mathbb{R}.$$

This is the desired conclusion. □

Corollary 1. *A Lebesgue measurable function $f : \mathbb{R} \longrightarrow \mathbb{R}$ yields a solution of the equation*

$$f(x)^2 + f(y)^2 + f(x+y)^2 - 2f(x)f(y)f(x+y) = 1$$

satisfied for all $x, y \in \mathbb{R}$, if and only if either $f(x) = -\frac{1}{2}$, $x \in \mathbb{R}$, or

$$f(x) = \begin{cases} 1 & for \ \ x \in S, \\ -\frac{1}{2} & for \ \ x \in S' \end{cases}$$

where $(S, +)$ is a Lebesgue measurable subgroup of the group $(\mathbb{R}, +)$ with $\ell_1(S) = 0$, or $f(x) = \cos bx$, $x \in \mathbb{R}$, or $f(x) = \cosh ax$, $x \in \mathbb{R}$, where $a, b \in \mathbb{R}$ are arbitrary constants.

Proof. Theorem 2 implies that in the case where card$f(\mathbb{R}) \geq 3$ one has

$$f(x) = \cos bx \cosh ax + i \sin bx \sinh ax, \ x \in \mathbb{R},$$

where $a, b \in \mathbb{R}$ stand for arbitrary constants. Since now f is a real function we conclude that either a or b has to vanish proving that either $f(x) = \cos bx$, $x \in \mathbb{R}$, or $f(x) = \cosh ax$, $x \in \mathbb{R}$, respectively. □

Corollary 2. (a solution of Cuculière's problem). *A continuous function* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *such that* $f(z) \leq 1$ *for some nonzero real number* z *yields a solution of the equation*

$$f(x)^2 + f(y)^2 + f(x+y)^2 - 2f(x)f(y)f(x+y) = 1$$

for all $x, y \in \mathbb{R}$, *if and only if either* $f(x) = -\frac{1}{2}$, $x \in \mathbb{R}$, *or* $f(x) = \cos bx$, $x \in \mathbb{R}$, *where* $b \in \mathbb{R}$ *is an arbitrary constant.*

Proof. Any continuous selfmapping of $\mathbb{R}$ is Lebesgue measurable and it suffices to apply Corollary 1: either $f(x) = -\frac{1}{2}$, $x \in \mathbb{R}$, or $f(x) = \cos bx$, $x \in \mathbb{R}$, or $f(x) = \cosh ax$, $x \in \mathbb{R}$, where $a, b \in \mathbb{R}$ are arbitrary constants, because the range of a continuous real function on $\mathbb{R}$ is connected. However, the cosh function is eliminated by the assumption that $f(z) \leq 1$ for some nonzero real number z. $\square$

4. An alternative equation

Without any regularity condition whatsoever, the facts established in the Introduction allow us to state the following

Remark. Given a group $(G, +)$ (not necessarily commutative), a function $f : G \longrightarrow \mathbb{C}$ such that $f(0) = 1$ yields a solution of the equation

$$f(x)^2 + f(y)^2 + f(x+y)^2 - 2f(x)f(y)f(x+y) = 1 \tag{C}$$

for all $x, y \in G$, then f satisfies an alternative d'Alembert functional equation

$$f(x+y) \neq f(x-y) \Longrightarrow f(x+y) + f(x-y) = 2f(x)f(y), \quad x, y \in G. \tag{CA}$$

So, in a natural way, we have entered the world of alternative functional equations like, for instance:

- Mikusiński's equation
$$f(x+y) \neq 0 \Longrightarrow f(x+y) = f(x) + f(y)$$
- the functional equation of Dhombres
$$f(x) + f(y) \neq 0 \Longrightarrow f(x+y) = f(x) + f(y)$$
- the Mikusiński–Pexider functional equation
$$f(x+y) \neq 0 \Longrightarrow g(x+y) = h(x) + h(y)$$
- numerous so called conditional equations

and so on (see e.g. Dhombres and Ger [3] or [4] for a more detailed review).

In that context, Eq. (CA) [stemming from Eq. (C) occurring in Roger Cuculière's problem] seems to be of interest on its own. It will be discussed in another paper.

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